XXXI. *Memoir on the Resultant of a System of two Equations.*

By ARTHUR CAYLEY, *Esq.*, *F.R.S.*

Received December 18, 1856,—Read January 8, 1857.

THE Resultant of two equations such as

$$(a, b, \ldots \between x, y)^m = 0,$$
$$(p, q, \ldots \between x, y)^n = 0,$$

is, it is well known, a function homogeneous in regard to the coefficients of each equation separately, viz. of the degree n in regard to the coefficients $(a, b, \ldots)$ of the first equation, and of the degree m in regard to the coefficients $(p, q, \ldots)$ of the second equation; and it is natural to develope the resultant in the form $k\mathrm{AP}+k'\mathrm{A'P'}+$ &c., where A, A′, &c. are the combinations (powers and products) of the degree n in the coefficients $(a, b, \ldots)$, P, P′, &c. are the combinations of the degree m in the coefficients $(p, q, \ldots)$, and k, k', &c. are mere numerical coefficients. The object of the present memoir is to show how this may be conveniently effected, either by the method of symmetric functions, or from the known expression of the Resultant in the form of a determinant, and to exhibit the developed expressions for the resultant of two equations, the degrees of which do not exceed 4. With respect to the first method, the formula in its best form, or nearly so, is given in the 'Algebra' of MEYER HIRSCH, and the application of it is very easy when the necessary tables are calculated: as to this, see my "Memoir on the Symmetric Functions of the Roots of an Equation*." But when the expression for the Resultant of two equations is to be calculated without the assistance of such tables, it is I think by far the most simple process to develope the determinant according to the second of the two methods.

Consider first the method of symmetric functions, and to fix the ideas, let the two equations be

$$(a, b, c, d \between x, y)^3 = 0,$$
$$(p, q, r \between x, y)^2 = 0.$$

Then writing

$$(a, b, c, d \between 1, z)^3 = a(1-\alpha z)(1-\beta z)(1-\gamma z),$$

so that

$$-\frac{b}{a} = \alpha+\beta+\gamma \qquad = (1),$$

$$+\frac{c}{a} = \alpha\beta+\alpha\gamma+\beta\gamma = (1^2),$$

$$-\frac{d}{a} = \alpha\beta\gamma \qquad = (1^3),$$

* Philosophical Transactions, 1857, pp. 489–497.

the Resultant is

$$(p,\ q,\ r\between\alpha,\ 1)^2.(p,\ q,\ r\between\beta,\ 1)^2.(p,\ q,\ r\between\gamma,\ 1)^2,$$

which is equal to

$$r^3+qr^2(\alpha+\beta+\gamma)+pr^2(\alpha^2+\beta^2+\gamma^2)+pqr(\alpha^2\beta+\alpha\beta^2+\beta^2\gamma+\beta\gamma^2+\gamma\alpha^2+\gamma^2\alpha)+\text{ \&c.}$$

Or adopting the notation for symmetric functions used in the memoir above referred to, this is

$$\begin{array}{l}\{\ \ r^3 \quad ,\\ \{+qr^2\ (1)\ ,\\ \left\{\begin{array}{l}+pr^2\ (2)\ ,\\ +q^2r\ (1^2)\ ,\end{array}\right.\\ \left\{\begin{array}{l}+pqr\ (21)\ ,\\ +q^3\ (1^3)\ ,\end{array}\right.\\ \left\{\begin{array}{l}+p^2r\ (2^2)\ ,\\ +pq^2\ (21^2),\end{array}\right.\\ \{+p^2q\ (2^21),\\ \{+p^3\ (2^3)\ ,\end{array}$$

the law of which is best seen by dividing by r^3 and then writing

$$\frac{q}{r}=[1],\quad \frac{p}{r}=[2],$$

and similarly,

$$\frac{q^2}{r^2}=[1^2],\quad \frac{pq}{r^2}=[21],\ \text{\&c.};$$

the expression would then become

$$1+[1](1)+[2](2)+[1^2](1^2)+[21](21)+[1^3](1^3)+[2^2](2^2)+[21^2](21^2)+[2^21](2^21)+[2^3](2^3),$$

where the terms within the [ ] and ( ) are simply all the partitions of the numbers 1, 2, 3, 4, 5, 6, the greatest part being 2, and the greatest number of parts being 3. And in like manner in the general case we have all the partitions of the numbers 1, 2, 3... mn, the greatest part being n, and the greatest number of parts being m.

The symmetric functions (1), (2), (1^2), &c. are given in the Tables (b) of the Memoir on Symmetric Functions, but it is necessary to remark that in the Tables the first coefficient a is put equal to unity, and consequently that there is a power of the coefficient a to be restored as a factor: this is at once effected by the condition of homogeneity. And it is not by any means necessary to write down (as for clearness of explanation has been done) the preceding expression for the Resultant; any portion of it may be taken out directly from one of the Tables (b). For instance, the bracketed portion

$$\begin{array}{l}+pqr\ (21),\\ +q^3\ \ (1^3),\end{array}$$

which corresponds to the partitions of the number 3, is to be taken out of the Table III(b). as follows: a portion of this Table (consisting as it happens of consecutive lines and columns, but this is not in general the case) is

=	d	bc
(21)	+3	−1
(1^3)	−1	

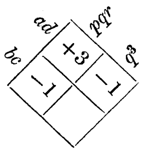
if in this we omit the sign =, and in the outside line write for homogeneity ad instead of d, and in the outside column first substituting q, p for 1, 2, then write for homogeneity pqr instead of pq, we have

	ad	bc
pqr	+3	−1
q^3	−1	

viz. $pqr\times(+3ad-1bc)+q^3(-1ad)$, for the value of the portion in question; this is equivalent to

	pqr	q^3
ad	+3	−1
bc	−1	

, or as it may be more conveniently written,

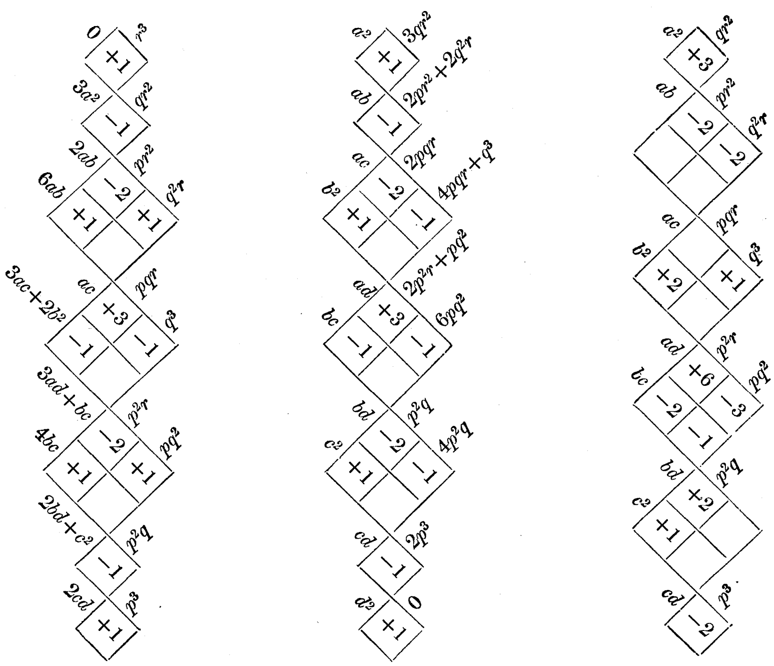


in which form it constitutes a part of the expression given in the sequel for the Resultant of the two functions in question; and similarly the remainder of the expression is at once derived from the Tables (b) I. to VI.

As a specimen of a mode of verification, it may be remarked that the Resultant qua invariant ought, when operated upon by the sum of the two operations,

$$3a\partial_b+2b\partial_c+c\partial_d \text{ and } 2p\partial_q+q\partial_r,$$

to give a result zero. The results of the two operations are originally obtained in the forms in the first and second columns, and the first column, and the second column, with all the signs reversed, are respectively equal to the third column, and consequently the sum of the first and second columns vanishes, as it ought to do.



Next to explain the second method, viz. the calculation of the resultant from the expression in the form of a determinant.

Taking the same example, as before, the resultant is

$$\left|\begin{array}{ccccc} & a, & b, & c, & d \\ a, & b, & c, & d, & \\ & & p, & q, & r \\ & p, & q, & r & \\ p, & q, & r, & & \end{array}\right|$$

which may be developed in the form

$$\begin{array}{l} +12.345\} \\ -13.245\} \\ \left.\begin{array}{l}+14.235 \\ +23.145\end{array}\right\} \\ \left.\begin{array}{l}-15.234 \\ -24.135\end{array}\right\} \\ \left.\begin{array}{l}+25.134 \\ +34.125\end{array}\right\} \\ -35.124\} \\ +45.123\} \end{array}$$

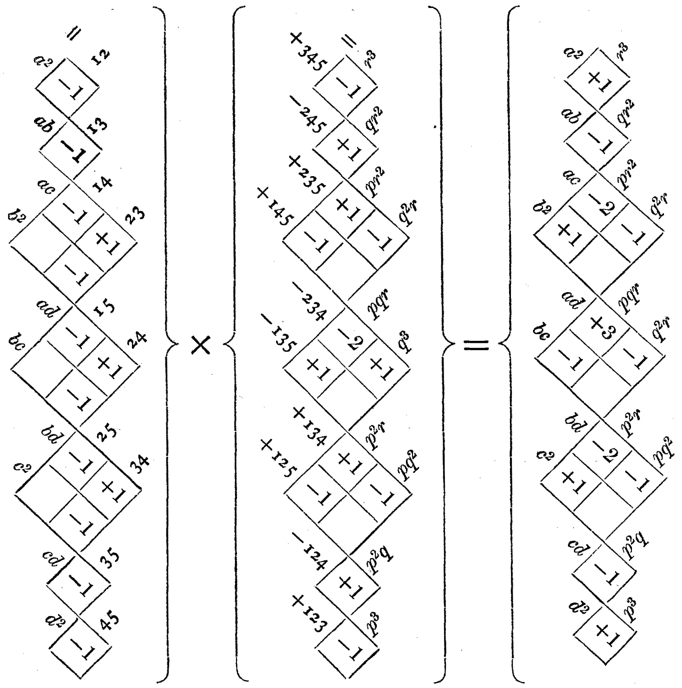
where 12, 13, &c. are the terms of

$$\left(\begin{array}{ccccc} & a, & b, & c, & d \\ a, & b, & c, & d & \end{array}\right)$$

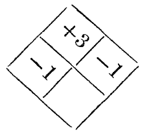
and 123, &c. are the terms of

$$\left(\begin{array}{ccccc} & & p, & q, & r \\ & p, & q, & r & \\ p, & q, & r & & \end{array}\right)$$

viz. 12 is the determinant formed with the first and second columns of the upper matrix, 123 is the determinant formed with the first, second and third columns of the lower matrix, and in like manner for the analogous symbols. These determinants must be first calculated, and the remainder of the calculation may then be arranged as follows:—



where it is to be observed that the figures in the squares of the third column are obtained from those in the corresponding squares of the first and second columns by the ordinary rule for the multiplication of determinants,—taking care to multiply the dexter lines (*i. e.* lines in the direction ╲) of the first square by the sinister lines (*i. e.* lines in the direction ╱) of the second square in order to obtain the sinister lines of the third square. Thus, for instance, the figures in the square



are obtained as follows, viz. the first sinister line $(+3,\ -1)$ by

$$(-1,\ +1)(-2,\ +1)=\ \ 2+1=+3$$
$$(-1,\ +1)(+1,\ \ 0)=-1+0=-1,$$

and the second sinister line $(-1,\ 0)$ by

$$(0,\ -1)(-2,\ +1)=0-1=-1$$
$$(0,\ -1)(+1,\ \ 0)=0+0=\ \ 0.$$

I have calculated the determinants required for the calculation, by the preceding process, of the Resultant of two quartic equations, and have indeed used them for the verification of the expression as found by the method of symmetric functions; as the determinants in question are useful for other purposes, I think the values are worth preserving.

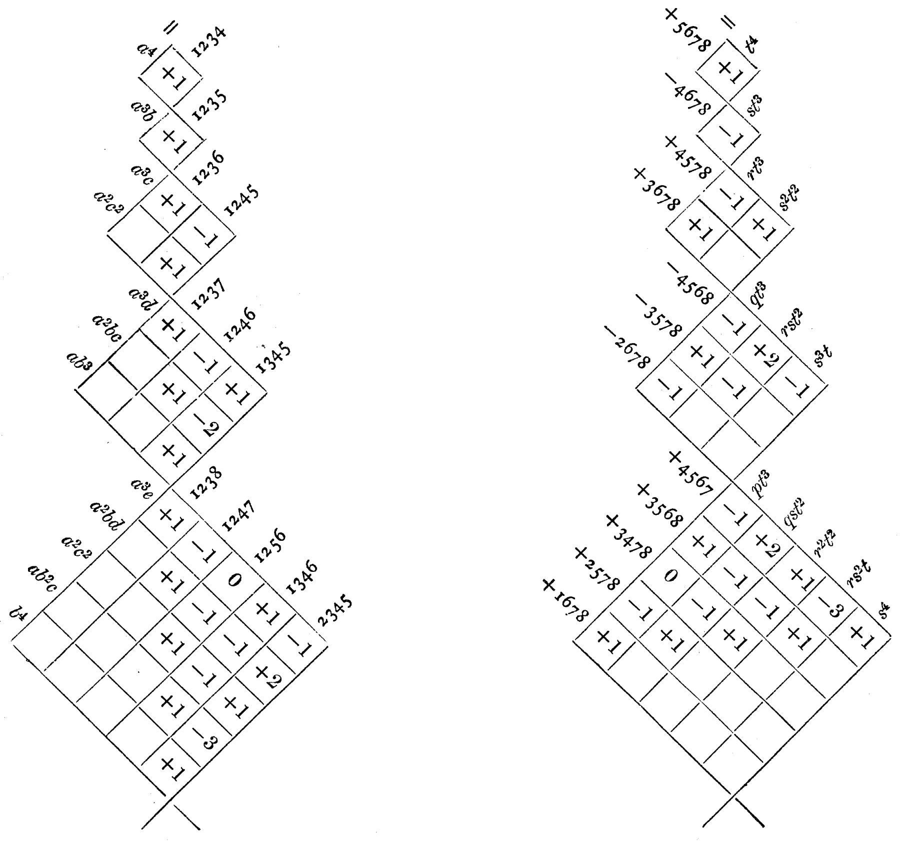
Table of the Determinants of the Matrices,

$$\left(\begin{array}{cccccccc} & & & a, & b, & c, & d, & e \\ & & a, & b, & c, & d, & e & \\ & a, & b, & c, & d, & e & & \\ a, & b, & c, & d, & e & & & \end{array}\right)$$

and

$$\left(\begin{array}{cccccccc} & & & p, & q, & r, & s, & t \\ & & p, & q, & r, & s, & t & \\ & p, & q, & r, & s, & t & & \\ p, & q, & r, & s, & t & & & \end{array}\right)$$

arranged in the form adapted for the calculation of the Resultant of the two quartic equations $(a, b, c, d, e \between x, y)^4=0$, and $(p, q, r, s, t \between x, y)^4=0$, viz.



	1248	1257	1347	1356	2346
a^2be	+1	−1	−1	+1	+1
a^2cd		+1	−1	−1	+2
ab^2d			+1	−1	−1
abc^2				+1	−2
b^3c					+1

	pst^2	qrt^2	qs^2t	r^2st	rs^3
−3567	−1	−2	+1	+2	−1
−3468	−1	+1	+1	−1	
−2568	+1	+1	−1		
−2478	+1	−1			
−1578	−1				

	1258	1348	1267	1357	2347	1456	2356
a^2ce	+1	−1	−1	0	+1	−1	+1
ab^2e		+1	0	−1	−1	+1	+1
a^2d^2			+1	−1	+1	+1	0
$abcd$				+1	−2	−2	0
b^3d					+1	0	−1
ac^3						+1	−1
b^2c^2							+1

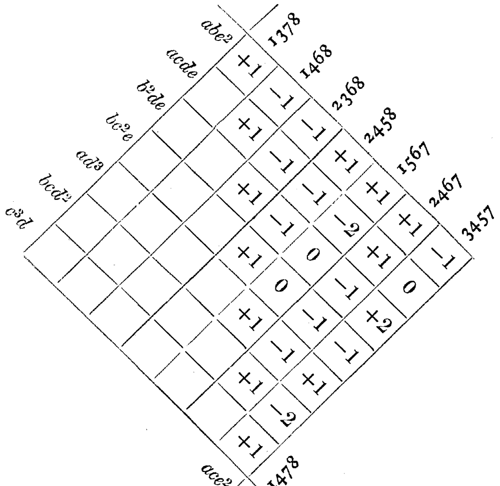
	prt^2	q^2t^2	ps^2t	$qrst$	r^3t	qs^3	r^2s^2
3467	+1	0	+1	0	−1	−1	+1
2567	+1	+1	−1	+2	0	+1	
3458	−1	+1	+1	−2	+1		
2468	0	−1	−1	+1			
1568	−1	0	+1				
2378	−1	+1					
1478	+1						

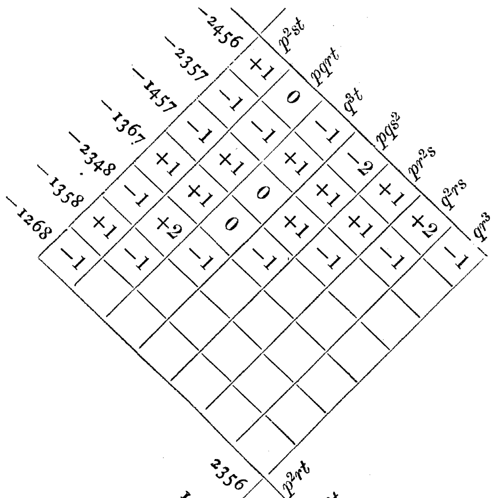
	1268	1358	2348	1367	1457	2357	2456
a^2de	+1	−1	+1	−1	+1	+1	−1
$abce$		+1	−2	−1	−1	+1	0
b^3e			+1	0	0	−1	+1
abd^2				+1	−1	−1	+2
ac^2d					+1	−1	−1
b^2cd						+1	−2
bc^3							+1

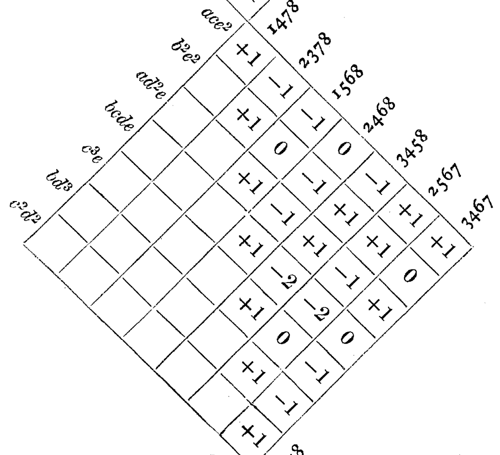
	pqt^2	$prst$	q^2st	qrt^3	ps^3	qrs^2	r^3s
−3457	+1	0	−2	+1	−1	+2	−1
−2467	−1	−1	+1	+1	+1	−1	
−1567	−1	+2	0	0	−1		
−2458	−1	+1	+1	−1			
−2368	+1	+1	−1				
−1468	+1	−1					
−1378	−1						

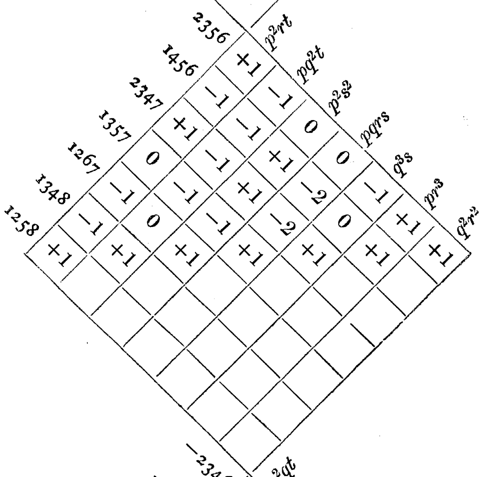
	1278	1368	1458	2358	1467	2367	2457	3456
a^2e^2	+1	−1	0	+1	+1	0	−1	1
$abde$		+1	−1	−1	−1	−1	+3	−2
ac^2e			+1	−1	−1	+1	0	−2
b^2ce				+1	0	−1	−1	+2
acd^2					+1	−1	−1	+2
b^2d^2						+1	−1	+1
bc^2d							+1	−3
c^4								+1

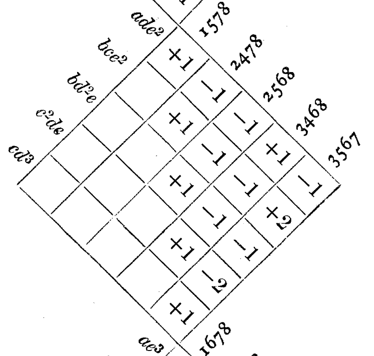
	p^2t^2	$pqst$	pr^2t	q^2rt	prs^2	q^2s^2	qr^2s	r^4
3456	+1	−2	−2	+2	+2	+1	−3	+1
2457	−1	+3	0	−1	−1	−1	+1	
2367	0	−1	+1	−1	−1	+1		
1467	+1	−1	−1	0	+1			
2358	+1	−1	−1	+1				
1458	0	−1	+1					
1368	−1	+1						
1278	+1							

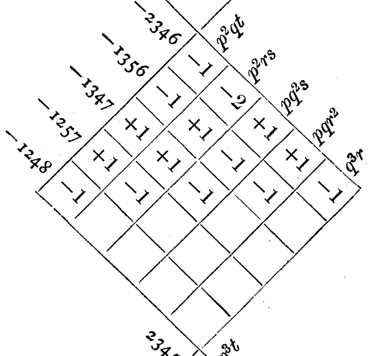

abe^2
$acde$
b^2de
bc^2e
ad^3
bcd^2
c^3d
1378
1468
2368
2458
1567
2467
3457

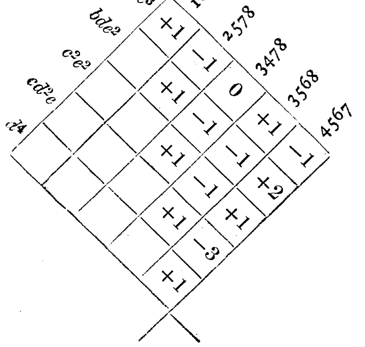

-2456
-2357
-1457
-1367
-2348
-1358
-1268
p^2st
$pqrt$
q^3t
pqs^2
pr^2s
q^2rs
qr^3

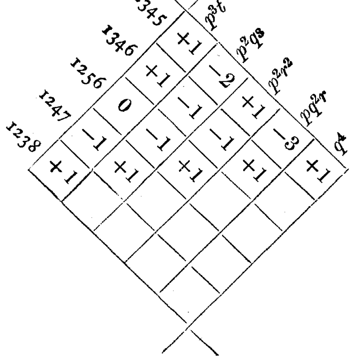

ace^2
b^2e^2
ad^2e
$bcde$
c^3e
bd^3
c^2d^2
1478
2378
1568
2468
3458
2567
3467

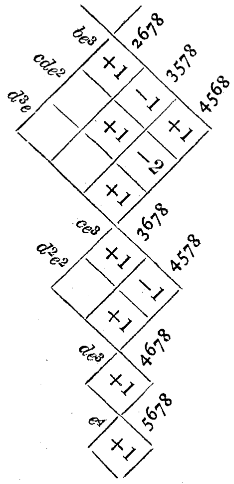

2356
1456
2347
1357
1267
1348
1258
p^2rt
pq^2t
p^2s^2
$pqrs$
q^3s
pr^3
q^2r^2

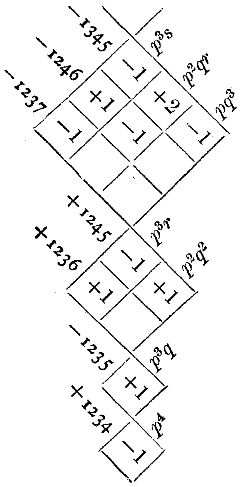

ade^2
bce^2
bd^2e
c^2de
cd^3
1578
2478
2568
3468
3567


-2346
-1356
-1347
-1257
-1248
p^2qt
p^2rs
pq^2s
pqr^2
q^3r


ae^3
bde^2
c^2e^2
cd^2e
d^4
1678
2578
3478
3568
4567


2345
1346
1256
1247
1238
p^3t
p^2qs
p^2r^2
pq^2r
q^4





The Tables of the Resultants of two equations which I have calculated are as follows:—

Table (2, 2).

Resultant of

$(a, b, c \between x, y)^2$,

$(p, q, r \between x, y)^2$.

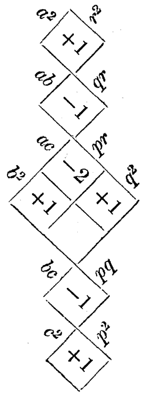


Table (3, 2).

Resultant of

$(a, b, c\ d \between x, y)^3$,

$(p, q, r \between x, y)^2$.

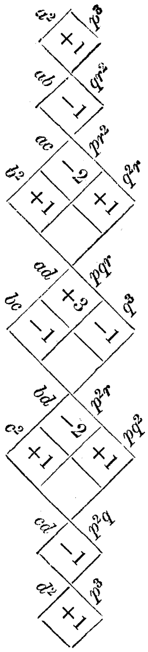


Table (4, 2).

Resultant of

$(a, b, c, d, e \between x, y)^4,$

$(p, q, r \between x, y)^2.$

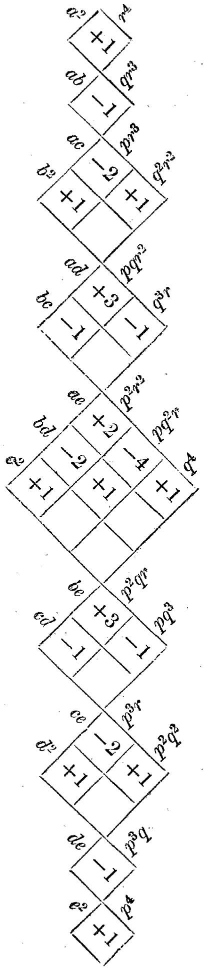


Table (3, 3).

Resultant of

$(a, b, c, d \between x, y)^3,$

$(p, q, r, s \between x, y)^3.$

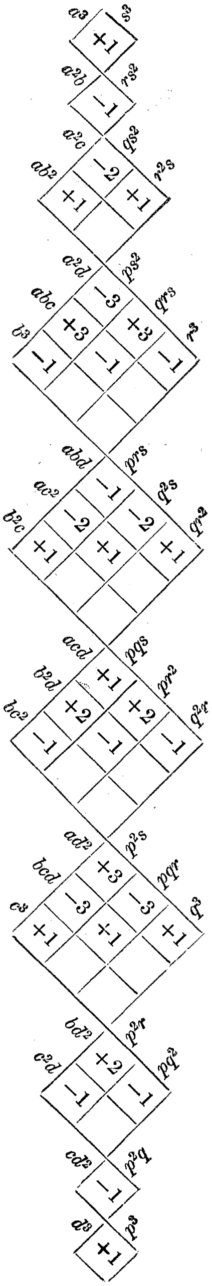


Table (4, 3).

Resultant of

$(a, b, c, d, e \between x, y)^4$,

$(p, q, r, s \between x, y)^3$.

	s^4
a^3	+1

	rs^3
a^2b	−1

	qs^3	r^2s^2
a^2c	−2	+1
ab^2	+1	

	ps^3	qrs^2	r^3s
a^2d	−3	+3	−1
abc	+3	−1	
b^3	−1		

	prs^2	q^2s^2	qr^2s	r^4
a^2e	+4	+2	−4	+1
abd	−1	−2	+1	
ac^2	−2	+1		
b^2c	+1			

	pqs^2	pr^2s	q^2rs	qr^3
abe	−5	+1	+3	−1
acd	+1	+2	−1	
b^2d	+2	−1		
bc^2	−1			

	p^2s^2	$pqrs$	q^3s	pr^3	q^2r^2
ace	−3	+4	−2	−2	+1
b^2e	+3	−3	0	+1	
ad^2	+3	−3	+1		
bcd	−3	+1			
c^3	+1				

	p^2rs	pq^2s	pqr^2	q^3r
ade	−5	+1	+3	−1
bce	+1	+2	−1	
bd^2	+2	−1		
c^2d	−1			

	p^2qs	p^2r^2	pq^2r	pq^3
ae^2	+4	+2	−4	+1
bde	−1	−2	+1	
c^2e	−2	+1		
cd^2	+1			

	p^3s	p^2qr	pq^3
be^2	−3	+3	−1
cde	+3	−1	
d^3	−1		

	p^3r	p^2q^2
ce^2	−2	+1
d^2e	+1	

	p^3q
de^3	−1

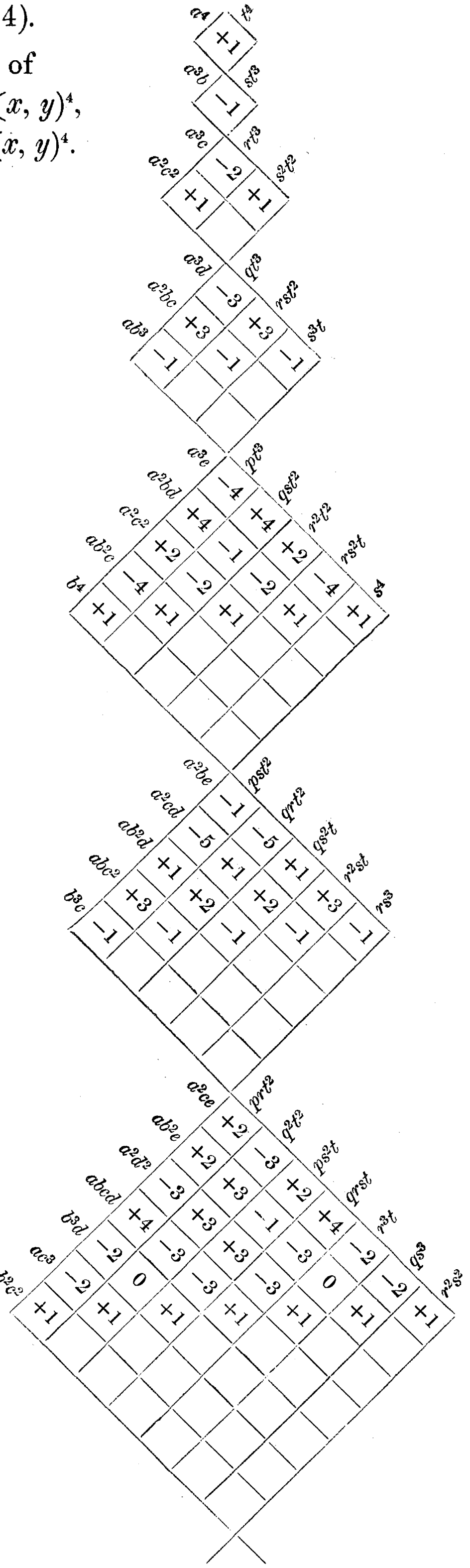
	p^4
e^4	+1

Table (4, 4).

Resultant of

$(a, b, c, d, e \between x, y)^4$,

$(p, q, r, s, t \between x, y)^4$.



	pqt^2	$prst$	q^2st	qr^2t	ps^3	qrs^2	r^3s
a^2de	+5	+2	−5	+1	−3	+3	−1
$abce$	+2	−8	+1	+2	+3	−1	
b^3e	−3	+3	0	0	−1		
abd^2	−5	+1	+2	−1			
ac^2d	+1	+2	−1				
b^2cd	+3	−1					
bc^3	−1						

	p^2t^2	$pqst$	pr^2t	q^2rt	prs^2	q^2s^2	qr^2s	r^4
a^2e^2	+6	−8	−4	+4	+4	+2	−4	+1
$abde$	−8	+10	0	−1	−1	−2	+1	
ac^2e	−4	0	+4	−2	−2	+1		
b^2ce	+4	−1	−2	0	+1			
acd^2	+4	−1	−2	+1				
b^2d^2	+2	−2	+1					
bc^2d	−4	+1						
c^4	+1							

	p^2st	$pqrt$	q^3t	pqs^2	pr^2s	q^2rs	qr^3
abe^2	+5	+2	−3	−5	+1	+3	−1
$acde$	+2	−8	+3	+1	+2	−1	
b^2de	−5	+1	0	+2	−1		
bc^2e	+1	+2	0	−1			
ad^3	−3	+3	−1				
bcd^2	+3	−1					
c^3d	−1						

	p^2rt	pq^2t	p^2s^2	$pqrs$	q^3s	pr^3	q^2r^2
ace^2	+2	+2	−3	+4	−2	−2	+1
b^2e^2	−3	+3	+3	−3	0	+1	
ad^2e	+2	−1	+3	−3	+1		
$bcde$	+4	−3	−3	+1			
c^3e	−2	0	+1				
bd^3	−2	+1					
c^2d^2	+1						

	p^2qt	p^2rs	pq^2s	pqr^2	q^3r
ade^2	−1	−5	+1	+3	−1
bce^2	−5	+1	+2	−1	
bd^2e	+1	+2	−1		
c^2de	+3	−1			
cd^3	−1				

	p^3t	p^2qs	p^2r^2	pq^2r	q^4
ae^3	−4	+4	+2	−4	+1
bde^2	+4	−1	−2	+1	
c^2e^2	+2	−2	+1		
cd^2e	−4	+1			
d^4	+1				

	p^3s	p^2qr	pq^3
be^3	−3	+3	−1
cde^2	+3	−1	
d^3e	−1		

	p^3r	p^2q^2
ce^3	−2	+1
d^2e^2	+1	

	p^3q
de^3	−1

	p^4
e^4	+1